Derived analytic geometry via entire functional calculus and dagger affinoids

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Entire functional calculus

K a complete valued field of characteristic 0.

Definition

$\mathcal{O}(K^n)$ the ring of holomorphic functions on K^n:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \lambda_{m_1,\ldots,m_n} z_1^{m_1} \cdots z_n^{m_n} \in K[\![z_1,\ldots,z_n]\!]$$

$$\lim_{\sum m_i \to \infty} |\lambda_{m_1,\ldots,m_n}|^{1/\sum m_i} = 0.$$

Definition

EFC K-algebra A: a commutative K-algebra with evaluations $f(a_1,\ldots,a_n) \in A$ for all $f \in \mathcal{O}(K^n)$.

Formally, A is a product-preserving functor

$$(\{K^n\}_n, \text{ analytic maps}) \to \text{Set}$$
$$K^n \mapsto A^n.$$

Examples

- commutative Banach K-algebras
- filtered limits of such, known as LMC (locally multiplicatively convex) topological commutative K-algebras
- hence Stein algebras
- quotients of these by any (not necessarily closed) ideal

Lemma

All finitely presented EFC algebras are Stein.

Proof.

$A = \mathcal{O}(K^n)/I$ for I finitely generated, so closed. □

(The topology on A is canonical.)

Lemma

EFC is the monad for the free-forget adjunction from commutative LMC algebras to sets.

Proof.

LMC completion of $K[S]$ is $\varinjlim_{\substack{T \subset S \\ \text{finite}}} \mathcal{O}(K^T)$, since

$$\varprojlim_{f\colon S \to \mathbb{R}_{>0}} \ell^1\langle \tfrac{x_s}{f(s)} \rangle_{s \in S} \cong \varinjlim_{\substack{T \\ \text{finite}}} \varprojlim_{r>0} \ell^1 \langle \tfrac{x_t}{r} \rangle_{t \in T}.$$

□

EFC-DGAs *(derived structures appear)*

Definition (Carchedi-Roytenberg)

An EFC-DGA A over K is a K-CDGA $A_\bullet$ with a compatible EFC structure on A_0 s.t. $\delta\colon A_0 \to A_{-1}$ is an EFC derivation.

We'll consider only $A = A_{\geq 0}$.

- Dold–Kan, with Eilenberg–Zilber shuffles gives:

Theorem (Nuiten)

Simplicial EFC-algebras and EFC-DGAs, localised at π_**- and* H_**-isomorphisms respectively, form equivalent* ∞*-categories.*

- Analogues for any Fermat theory (Hadamard's lemma $\frac{f(x,\underline{z})-f(y,\underline{z})}{x-y} \in \mathcal{O}(K^{n+2})\ \forall f \in \mathcal{O}(K^{n+1})$).

Model structures on EFC-DGAs $A_{\geq 0}$

- Weak equivalences of EFC-DGAs are H_*-isomorphisms.
- Standard model structure:
  - fibrations are surjective in positive degrees
  - f.g. cofibrant objects are retracts of $(\mathcal{O}(K^n)[y_1, \ldots, y_m], \delta)$, deg $y_i > 0$.
- Local model structure:
  - additional cofibrations gen'd by $\mathcal{O}(K^n) \to \mathcal{O}(U)$ for open Stein submanifolds $U \subset K^n$
  - fewer fibrations (RLP on $A_0 \to \mathrm{H}_0 A \times_{\mathrm{H}_0 B} B_0$).
- They are Quillen equivalent.

∞-equivalences

Theorem

The forgetful functor from LMC topological CDGAs $A_{\geq 0}$ to EFC-DGAs $A_{\geq 0}$ becomes an equivalence on ∞-localisation at abstract H^*-isomorphisms.

Proof.

The unit of the adjunction is an isomorphism on any cofibrant EFC-DGA. □

Corollary

A functor on LMC topological CDGAs descends to an ∞-functor on EFC-DGAs iff it sends abstract H^*-isomorphisms to weak equivalences.

(Good for maps between analytifications.)

Cotangent complexes (Quillen's theory)

- Ω^1_A the A-module representing the functor

$$M \mapsto \mathrm{Hom}_{EFC}(A, A \oplus M\epsilon) \times_{\mathrm{Hom}_{EFC}(A,A)} \{\mathrm{id}\}$$

- $\mathbb{L}\Omega^1_A := \Omega^1_{\tilde{A}} \otimes_{\tilde{A}} A$, for $\tilde{A} \to A$ cofibrant replacement
- H_0 and $\mathbb{L}\Omega^1$ detect H_*-isomorphisms
- $\mathbb{L}\Omega^1_{B/A}$ is alg. cotangent complex if $A_0 \cong B_0$
- Local model structure suffices, so

$$\mathbb{L}\Omega^1_{\mathcal{O}(U)} \simeq \Omega^1_{\mathcal{O}(U)} \simeq \Gamma(U, \Omega^1_U)$$

  for any Stein manifold U.
- $\rightsquigarrow$ symplectic and Poisson structures.

Comparison with Lurie/Porta–Yu I

Theorem

For Y and X (L/P–Y) derived K-analytic spaces s.t. classical locus t_0X is Stein (+mild finiteness),

$$\mathrm{map}_{\mathrm{dAn}_K}(Y,X) \xrightarrow{\sim} \mathrm{map}_{EFC}(\mathrm{R}\Gamma(X,\mathscr{O}_X),\mathrm{R}\Gamma(Y,\mathscr{O}_Y)).$$

Proof.

Work up the Postnikov tower of $\mathscr{O}_Y$, using Wiegmann's or Lütkebohmert's embedding theorem for $\mathrm{H}_0\mathscr{O}_X$, then cotangent complexes. □

- Essentially, X is a homotopy limit of spaces K^n, so most pregeometric data redundant here.

Dagger dg algebras

Definition

Washnitzer algebra $K\langle \frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n} \rangle^\dagger$ the Noetherian ring of overconvergent functions on a polydisc:

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \lambda_{m_1,\ldots,m_n} z_1^{m_1} \cdots z_n^{m_n} \in K[\![z_1, \ldots, z_n]\!]$$

$$|\lambda_{\underline{m}}|\rho_1^{m_1} \ldots \rho_n^{m_n} \xrightarrow{|\underline{m}|\to\infty} 0 \text{ for some } \rho_i > r_i.$$

Quotients of these are *dagger algebras*.

Definition

A dagger DGA $A_{\geq 0}$ is a K-CDGA with A_0 a dagger algebra and the A_0-modules A_m all finite.

Definition
A dagger DGA A is *quasi-free* if A_0 is isomorphic to a Washnitzer algebra and A is freely generated as a graded-commutative A_0-algebra.

- Every dagger algebra is naturally an ind-Banach algebra, and we then have:

Theorem
The forgetful functor from dagger DGAs to EFC-DGAs becomes ∞-fully faithful on inverting H_**-isomorphisms.*

Proof.
Noetherianity gives quasi-free cosimplicial frames for dagger DGAs. They yield mapping spaces via local model structure on EFC-DGAs, since cofibrant. □

Exact functors on dagger DGAs

- Dagger algebras are ind-(nuclear Fréchet), since

$$K\langle \frac{x_1}{r_1}, \ldots, \frac{x_n}{r_n} \rangle^\dagger = \varinjlim_{\rho_i > r_i} O(\Delta(\rho_1, \ldots, \rho_n)).$$

- Noetherianity makes $A_\bullet$ strictly exact, so:

Corollary

For dagger DGAs A and Fréchet spaces V we have $\mathrm{H}_*(A \hat{\otimes}_\pi V) \cong \mathrm{H}_*(A) \hat{\otimes}_\pi V$.

- $\rightsquigarrow$ many functors on corresponding EFC-DGAs, e.g for V a ring of continuous functions (inc. condensed) or smooth forms.

Dagger dg spaces

Definition

A K-dagger dg space X is pair $(\pi^0 X, \mathscr{O}_X)$ where:

- $\pi^0 X$ is a K-dagger space (Grosse–Klönne)
- $\mathscr{O}_X$ is a presheaf of dagger DGAs on its site of open affinoid subdomains
- $\mathrm{H}_0 \mathscr{O}_X = \mathscr{O}_{\pi^0 X}$
- $\mathrm{H}_i \mathscr{O}_X$ are all coherent $\mathscr{O}_{\pi^0 X}$-modules.

Example: $(\mathrm{Sp}(\mathrm{H}_0 A), \iota^{-1} A)$, for A a dagger DGA and $\iota \colon \mathrm{Sp}(\mathrm{H}_0 A) \to \mathrm{Sp}(A_0)$.

- $X \to Y$ a *weak equivalence* if $\pi^0 X \cong \pi^0 Y$ and $\mathrm{H}_* \mathscr{O}_X \cong \mathrm{H}_* \mathscr{O}_Y$.

Comparison with Lurie/Porta–Yu II

For p.p.= partially proper (≈ without boundary):

Theorem

The ∞-category of p.p. (L/P–Y) derived K-analytic spaces is equivalent to the ∞-category of dg dagger spaces X with $\pi^0 X$ p.p.

Proof.

Rigid analytic and dagger analytic spaces correspond when p.p. Cotangent complexes agree, so we induct on Postnikov tower of $\mathscr{O}_X$. □

Non-commutative analogues and challenges

- (Pirkovskii) Free EFC (FEFC) algebras based on completion $\mathcal{F}_n$ of tensor algebra.
- FEFC DGAs via graded completions of graded tensor algebra (higher degrees subtler than in commutative case).
- FEFC-DGAs Quillen equivalent to simplicial FEFC (uses convergence of homotopies).
- FEFC-DGAs equivalent to localising LDMC (i.e. pro-Banach) dg algebras at H_*-isos.
- On (nuclear) Fréchet DGAs, $-\hat{\otimes} V$ preserves H_*-isos for V)nuclear(Fréchet.

- Hard to find a good (∞-)subcategory of LDMC algebras:
- Fewer Noetherian algebras, so cofibrant replacement hard.
- Cotangent complexes liable to explode.
- Without Noetherianity, complexes aren't strict — few functors on H_*-localisation.
- Fewer nuclear algebras ($\mathcal{F}_n$ has n^m coefficients in degree m):
- on $\mathcal{F}_n$, the map between completions w.r.t. $\| - \|_\rho$ and $\| - \|_{\rho'}$ is only nuclear for $\rho' > n\rho$ (cf. $\rho' > \rho$ in commutative case).
- Hence only analogues of compact Steins or overconvergence are close to commutative.

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