Higher-categorical logarithmic structures from higher Brauer groups

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Invitation to derived geometry in Padova
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Review of classical log structures

Log structure on a stack X (Fontaine–Illusie style, [Kato])

Monoid $\mathcal{M}$ in $X_{\text{ét}}$ with $\mathcal{M} \xrightarrow{\alpha} \mathcal{O}_X$ such that $\alpha^{-1}\mathcal{O}_X^\times \xrightarrow{\simeq} \mathcal{O}_X^\times$

Homotopical version: spectral (affine) by [Rognes], derived by [Sagave–Schürg–Vezzosi]

Motivating example

D divisor on $X \rightsquigarrow$ log structure $\mathcal{O}_X(D) \to \mathcal{O}_X$

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Remark: Stack quotient by $\mathcal{O}_X^\times$ gives $\overline{\alpha}\colon \overline{\mathcal{M}} \coloneqq [\mathcal{M}/\mathcal{O}_X^\times] \to [\mathcal{O}_X/\mathcal{O}_X^\times]$

Stacky reformulation (Deligne–Faltings style, [Borne–Talpo–Vistoli])

Pre-log structure: monoid $\overline{\mathcal{M}}$ in $X_{\text{ét}}$ with $\overline{\mathcal{M}} \xrightarrow{\overline{\alpha}} [\mathbb{A}^1_X/\mathbb{G}_{\mathrm{m},X}] \eqqcolon \mathcal{C}art_X$

Log structure if $\overline{\alpha}$ has trivial kernel

Shifted divisors in derived geometry

Line bundles in derived geometry

- $\mathscr{P}\!ic_X = \mathcal{B}\,\mathbb{G}_{\mathrm{m}} = [*/\mathbb{G}_{\mathrm{m}}]$ moduli stack of invertible objects in $\mathfrak{QCoh}(X)^{\mathrm{cn}}$
- Invertible objects in $\mathfrak{QCoh}(X)$ classified by $\mathscr{P}\!ic^{\dagger}_X = \underline{\mathbb{Z}}_X \times \mathcal{B}\,\mathbb{G}_{\mathrm{m}}$: shifted line bundles

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What is the stack $\mathscr{C}art^{\dagger}_X$ of "shifted divisors"?

Idea: $$\text{“}\coprod_{n \in \underline{\mathbb{Z}}_X}\text{”}\left[\mathbb{A}^1[n]/\mathbb{G}_{\mathrm{m}}\right]$$

where "$\coprod_{n \in \underline{\mathbb{Z}}_X}$" means: on each connected component U, take $\coprod_{n \in \mathbb{Z}} \left[\mathbb{A}^1_U[n]/\mathbb{G}_{\mathrm{m},U}\right]$

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What is the stack $\mathscr{Cart}_X^{\dagger}$ of "shifted divisors"?

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Problem:
- How to make sense formally of this sheaf-indexed colimit?
- What is the "induced" monoid structure on $\mathscr{Cart}_X^{\dagger}$?

Toward internal colimits

Observation: G a group, $\alpha\colon \mathfrak{B}G \to \mathfrak{C}$ an object C with G-action in a category $\mathfrak{C}$
Then $\varprojlim_{\mathfrak{B}G} \alpha = C^G$ object of G-invariants, and $\underrightarrow{\mathrm{colim}}_{\mathfrak{B}G}\, \alpha = C_G = C/G$ quotient

What to do for G an algebraic group over some X, *i.e.* internal group in $X_{\text{ét}}$?

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Two incarnations of $\mathcal{B}G$

- $\mathcal{B}\,G = [*/G]$ object of the $(\infty, 1)$-topos $X_{\text{ét}} \simeq \mathfrak{Shv}_{\infty\text{-}\mathfrak{Grpd}}(X_{\text{ét}})$
- $B_\bullet G = \ker(* \to \mathcal{B}\,G)$ internal groupoid in $X_{\text{ét}}$, with $B_n G = G^n$

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Two incarnations of internal categories in an $(\infty,1)$-topos $\mathfrak{X}$

- Category object in $\mathfrak{X}$ is $C_\bullet\colon \Delta^{\mathrm{op}} \to \mathfrak{X}$ with Segal condition $C_n \xrightarrow{\simeq} C_1 \times_{C_0} \cdots \times_{C_0} C_1$
  Internal category if it is Rezk-complete, aka univalent
- Sheaf of $(\infty,1)$-categories on $\mathfrak{X}$ (object of the $(\infty,2)$-topos $\mathfrak{Shv}_{(\infty,1)\text{-}\mathfrak{Cat}}(\mathfrak{X})$)

Internal colimits

Internal category theory in $\mathfrak{X}$ $=$ "category theory in the internal language of $\mathfrak{X}$"
$=$ formal category theory in $\mathfrak{Cat}(\mathfrak{X})$

Internal colimits [Martini–Wolf]

$\mathscr{I}, \mathscr{C}$ internal categories. $\exists$ internal functor $\Delta\colon \mathscr{C} \to \mathscr{C}^{\mathscr{I}}$, "constant diagrams"

- $\mathscr{C}^{\mathscr{I}}$ is the exponential, $\mathfrak{X} \ni Z \mapsto \mathfrak{Fun}(Z \times \mathscr{I}, \mathscr{C})$

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Quotients of internal group actions

- Any $(\infty, 1)$-topos $\mathfrak{X}$ has an object classifier Ω, such that $\hom(Z, \Omega) \simeq (\mathfrak{X}_{/Z})^{\simeq}$

$$\implies \quad \Omega \text{ is the core of an } \mathfrak{X}\text{-category } \mathscr{U}\colon Z \mapsto \mathfrak{X}_{/Z}, \text{ the } \textbf{universe}$$

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  $\implies$ $\varinjlim_{\mathcal{B}G}$ is left-adjoint to $(-) \times \mathcal{B}G\colon \mathfrak{X}_{/*} \to \mathfrak{X}_{/\mathcal{B}G}$: quotient $[-/G]$

Upshot: The stack of shifted divisors as an internal colimit

Total space of line bundles

Internal functor $\Omega^\infty\colon \mathcal{B}\,\mathbb{G}_{m,X} = \mathscr{P}ic_X \to \mathscr{U}$ by:

$$\mathscr{P}ic_X(T) = \{\text{line bundles on } T\} \ni \mathscr{L} \mapsto \Omega^\infty \mathscr{L} = \mathbb{V}_T(\mathscr{L}) \in \mathfrak{dSt}_{/T} = \mathscr{U}(T)$$

By before: $\mathscr{C}art_X \coloneqq \left[\mathbb{A}^1_X/\mathbb{G}_{m,X}\right] = \varinjlim_{\mathscr{P}ic_X} \Omega^\infty$

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- $\Omega^\infty \colon \mathscr{P}\!ic^\dagger_X \simeq \mathbb{Z}_X \times \mathscr{P}\!ic_X \to \mathcal{U}, (n, \mathscr{L}) \mapsto \Omega^\infty(\mathscr{L}[n])$

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- $\mathcal{C}art^\dagger_X := \varinjlim_{\mathcal{P}ic^\dagger_X} \Omega^\infty$ (think $\varinjlim_{n \in \underline{\mathbb{Z}}_X} \varinjlim_{\mathbb{A}^1 \in \mathcal{B}\,\mathbb{G}_m} \Omega^\infty(\mathbb{A}^1[n])$)

Gradings

By "Fubini", we can also think of $\mathscr{C}art^{\dagger}_X$ as $\left[\coprod_{n\in\mathbb{Z}_X} \Omega^\infty(\mathbb{A}^1[n]) \Big/ \mathbb{G}_{\mathrm{m}}\right]$

Total shift of $M \in \mathfrak{QCoh}(X)$: $\Omega^\infty_{\mathrm{gr}} M = \coprod_{n\in\mathbb{Z}} \Omega^\infty(M[n])$

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Interpretation of the total shift

- Give M chaotic $\mathbb{Z}$-grading: $(M^{\mathrm{chs}})_n = M$ component $\forall n \in \mathbb{Z}$

Remark: Groupoid-indexed colimits as gradings:

$$\begin{array}{ccc} \mathcal{U}^{\mathcal{I}} & \xrightarrow{\;\simeq\;} & \mathcal{U}_{/|\mathcal{I}|} \\ & \underset{\xrightarrow{\mathrm{colim}}_{\mathcal{I}}}{\searrow} \quad \swarrow & \\ & \mathcal{U} & \end{array}$$

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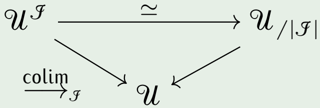

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$$\mathcal{U}^{\mathcal{I}} \xrightarrow{\ \simeq\ } \mathcal{U}_{/|\mathcal{I}|}, \qquad \mathcal{U}^{\mathcal{I}} \xrightarrow{\operatorname{colim}_{\mathcal{I}}} \mathcal{U}, \qquad \mathcal{U}_{/|\mathcal{I}|} \to \mathcal{U}$$

Problem: $\mathbb{Z}$-shearing is only $\mathcal{E}_2$- but not $\mathcal{E}_3$-monoidal over $\mathbb{S}$ [Lurie]

Graded infinite loop spaces

Shift is an automorphism of $\mathfrak{QCoh}(X) = \Gamma(\mathscr{QCoh}_X)$: monoidal functor $\underline{\mathbb{Z}}_X \to \mathscr{End}\left(\mathscr{QCoh}_X\right)$

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Equivalence of internal $\mathcal{E}_1$-monoidal categories $\mathscr{End}^{\mathrm{ex}}(\mathscr{QCoh}_X) \simeq \mathscr{QCoh}_X$

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Shifting extends to an $\mathbb{S}$-automorphism $\underline{\Omega^\infty\mathbb{S}}_X \to \mathscr{End}(\mathscr{QCoh}_X)^\times$

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$$\implies \mathscr{Cart}^\$_X = \underset{\overrightarrow{\mathscr{Pic}^\$_X}}{\operatorname{colim}} \Omega^\infty \simeq [\Omega^\infty_\$ \mathbb{A}^1_X / \mathbb{G}_{\mathrm{m},X}]$$

The higher Brauer groups

Presentable stable (∞, n)-categories [Stefanich]

- A (presentable) stable $(\infty, 0)$-category is a spectrum
- A presentable stable $(\infty, n+1)$-category is a compact object in $\mathfrak{St}_n$-$\mathfrak{Cat}$

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A $\mathcal{E}_\infty$-ring $\rightsquigarrow$ $A\text{-}\mathfrak{Mod}_n \coloneqq \left(A\text{-}\mathfrak{Mod}_{n-1}\right)\text{-}\mathfrak{Mod}$ an $\mathcal{E}_\infty$-monoidal stable (∞, n)-cat

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Lemma [Stefanich]

$A \mapsto A\text{-}\mathfrak{Mod}_n$ satisfies étale descent $\rightsquigarrow$ sheaf of (∞, n)-cats $\mathfrak{QCoh}_n$

We will define $\mathscr{Br}_n^{\$} = \mathfrak{QCoh}_{n+1}^{\times}$

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We will define $\mathscr{Br}^{\$}_n = \mathfrak{QCoh}^{\times}_{n+1}$ (Conjecture: It recovers Haugseng's Brauer groups)

Low ns: $\mathscr{Br}^{\$}_{-1} = \mathbb{G}_{\mathrm{m}}$ (connectively), $\mathscr{Br}^{\$}_0 = \mathscr{Pic}^{\$}$, and $\mathscr{Br}^{\$}_1$ is the extended Brauer stack

Higher affines

n-categorical structure sheaf $\mathscr{QCoh}_n \in \mathfrak{QCoh}_{n+1}(X)$, with $\Gamma(X, \mathscr{QCoh}_n) = \mathfrak{QCoh}_n(X)$

n-affine stacks [Stefanich, Gaitsgory for $n = 1$]

A derived stack X is n-affine if $\Gamma(X, -) \colon \mathfrak{QCoh}_{n+1}(X) \xrightarrow{\simeq} \mathfrak{QCoh}_n(X)\text{-}\mathfrak{Mod}$

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- An n-categorical ring is a symmetric monoidal stable (∞, n)-category
- $(\infty, n+1)$-category of n-affines $\mathfrak{Aff}_n = \mathfrak{Ring}_n{}^{\mathrm{op}}$

n-categorical stacks: functors $\mathfrak{Aff}_n \to (\infty, n)\text{-}\mathfrak{Cat}$ with n-categorical descent

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n-categorical stacks: functors $\mathfrak{Aff}_n \to (\infty, n)\text{-}\mathfrak{Cat}$ with n-categorical descent

Embedding $\imath_{n,k} \colon \mathfrak{Aff}_n \hookrightarrow \mathfrak{Aff}_{n+k}$, $A \mapsto A\text{-}\mathfrak{Mod}_k$ inducing $\imath^*_{n,k} \colon (n+k)\text{-}\mathfrak{dSt} \to n\text{-}\mathfrak{dSt}$

A derived stack is n-affine iff it is of the form $\imath^*_{0,n}\,\mathrm{Spec}(A)$ for A an n-ring

Higher-categorical logarithmic structures

- In the $(\infty, n+1)$-topos $n\text{-}\mathfrak{dSt}_{/X}$, internal n-functor

$$\tau\colon \mathscr{Br}_n^{\$} \to \mathscr{U} \quad \text{(underlying } (\infty, n)\text{-category of a stable } (\infty, n)\text{-cat)}$$

- n-categorical (extended) Cartier stack: $\mathscr{Cart}_n^{\$} = \varinjlim_{\mathscr{Br}_n^{\$}} \tau$

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Expectations

- The infinite root stack of an n-categorical log structure will be an n-stack
- For n-log structures on curves, recover higher gerbes $\mathcal{B}^n \mu_\infty$